Serre Spectral Sequence

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1 Definitions

This completes the construction that we motivated spectral sequences with. Consider a Serre fiber sequence $F \to X \xrightarrow{\pi} B$ where B is a finite CW complex and F is path connected. Then we get a family of fiber sequences by restricting the fibration

$$X_p \ := \ \pi^{-1}(B_p) \subseteq X \to B_p$$

It is a fact that these maps are still fibrations. The previous case is for the trivial fibration $B \to B$. We claim that the following is an exact couple

$$\begin{tikzcd}
\bigoplus H^q(X_p) \arrow[rr, "\mathrm{res} X_{p-1}"] & & \bigoplus H^q(X_p) \arrow[dl, "\partial"] \\
& \bigoplus H^q(X_p, X_{p-1}) \arrow[ul, "\mathrm{res} X_p"] &
\end{tikzcd}$$

Where ∂ is the coboundary map from the LES of pairs, and res stands for the restriction of a map to the given set. The fact that this is an exact couple is clear because we have just "rolled up" the LES in cohomology for pairs. The Serre SS has also the simplifying characteristics

Theorem (Hatcher Chapter 5, Thm 5.3, [**?**] Thm 5.3.2, Example 5.2.2)**.** *If the base is simply connected then*

- *The second page is characterised by* $E^2_{pq} \cong H_p(B; H_q(F))$.
- *If the fibration splits, that is* $X = B \times F$ *then the SSS degenerates at page 2.*
- *The sequence is first quadrant.*
- *The sequence converges to* $H^*(X)$*, more precscicely we have that when* $p + q = n$ *then* $E^{\infty}_{p,q} = H^n(X)$

- *The stable terms $E^{\infty}_{p,n-p}$ are isomorphic to the quotients F^p_n/F^{p-1}_n for some filtration F^p_n of $H^n(X)$. (The infinity term is here just the stable values as the sequence is first quadrant it degenerates at a finite page).*

If we perform the re-indexing given by $q \mapsto p+q$ then we again have a spectral sequence with differentials now
$$d^{pq}_r : E^{pq}_r \to E^{p+r,q+r-1}_r$$
which still go straight across as desired, however the groups are given by
$$E^{pq}_1 = H^{p+q}(X_p, X_{p-1})$$
Notice that this also brings us into complete parity with the initial sketch / motivation that we provided. Note that this reindexing is a visual change, studying the maps specified by the exact couple we require the maps to change degree in this way, the point is that the exact couple is *exact*, and so the maps are not those that are given by just going around the triangle, we must do what is specified here.

2 Complex Projective Spaces

Here we will use the SSS to compute the cohomology of $\mathbb{C}P^n$. We will consider the fibration
$$* \to \mathbb{C}P^n \to \mathbb{C}P^n$$
And recall that $\mathbb{C}P^n$ has a cellular structure given by
$$\emptyset \subseteq * \subseteq \mathbb{C}P^1 \subseteq \cdots \subseteq \mathbb{C}P^{n-1} \subseteq \mathbb{C}P^n$$
Then right away using the results on SSS we know that the E^2 page is given by

$$\begin{array}{cccc}
\vdots & \vdots & \vdots & \vdots \\
H^0(\mathbb{C}P^n; H^1(*)) & H^1(\mathbb{C}P^n; H^1(*)) & H^2(\mathbb{C}P^n; H^1(*)) & H^3(\mathbb{C}P^n; H^1(*)) \\
H^0(\mathbb{C}P^n; H^0(*)) & H^1(\mathbb{C}P^n; H^0(*)) & H^2(\mathbb{C}P^n; H^0(*)) & H^3(\mathbb{C}P^n; H^0(*))
\end{array}$$

Using that the cohomology of a point is just $\mathbb{Z}$ in degree 0 and 0 elsewhere this is then

$$\begin{array}{cccc}
\vdots & \vdots & \vdots & \vdots \\
H^0(\mathbb{C}P^n; 0) & H^1(\mathbb{C}P^n; 0) & H^2(\mathbb{C}P^n; 0) & H^3(\mathbb{C}P^n; 0) \\
H^0(\mathbb{C}P^n; \mathbb{Z}) & H^1(\mathbb{C}P^n; \mathbb{Z}) & H^2(\mathbb{C}P^n; \mathbb{Z}) & H^3(\mathbb{C}P^n; \mathbb{Z})
\end{array}$$

Or simply

$$
\begin{array}{cccc}
\vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 \\
H^0(\mathbb{C}P^n) & H^1(\mathbb{C}P^n) & H^2(\mathbb{C}P^n) & H^3(\mathbb{C}P^n)
\end{array}
$$

Thus this SS clearly both degenrates and collapses at (the latest at) the E^2 page. We also know that these groups are isomorphic to the kernel mod the image of the groups on the E^1 page, using our formula we can write out the E^1 page as

$$
\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \\
H^2(\mathbb{C}P^0, \mathbb{C}P^{-1}) & \longrightarrow & 0 & \longrightarrow & H^4(\mathbb{C}P^1, \mathbb{C}P^0) & \longrightarrow & 0 & \longrightarrow & H^6(\mathbb{C}P^2, \mathbb{C}P^1) & \cdots \\
H^1(\mathbb{C}P^0, \mathbb{C}P^{-1}) & \longrightarrow & 0 & \longrightarrow & H^3(\mathbb{C}P^1, \mathbb{C}P^0) & \longrightarrow & 0 & \longrightarrow & H^5(\mathbb{C}P^2, \mathbb{C}P^1) & \cdots \\
H^0(\mathbb{C}P^0, \mathbb{C}P^{-1}) & \longrightarrow & 0 & \longrightarrow & H^2(\mathbb{C}P^1, \mathbb{C}P^0) & \longrightarrow & 0 & \longrightarrow & H^4(\mathbb{C}P^2, \mathbb{C}P^1) & \cdots
\end{array}
$$

Noting that for p odd $\mathbb{C}P^n$ has no p skeleton. We claim that

$$
H^a(\mathbb{C}P^b, \mathbb{C}P^{b-1}) = H^a(S^{2b}) = \begin{cases} \mathbb{Z}, & a = 0, 2b \\ 0, & \text{else} \end{cases}
$$

as the CW structure is gluing a D^{2n} into the previous C^{n-1} and so when we collapse that we just get the $2n$ sphere. Plugging these values into the spectral sequence gives

$$
\begin{array}{cccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & & \\
H^2(S^0) & \longrightarrow & 0 & \longrightarrow & H^4(S^2) & \longrightarrow & 0 & \longrightarrow & H^6(S^4) & \cdots \\
H^1(S^0) & \longrightarrow & 0 & \longrightarrow & H^3(S^2) & \longrightarrow & 0 & \longrightarrow & H^5(S^4) & \cdots \\
H^0(S^0) & \longrightarrow & 0 & \longrightarrow & H^2(S^2) & \longrightarrow & 0 & \longrightarrow & H^4(S^4) & \cdots
\end{array}
$$

or explicitly

$$\begin{array}{cccc}
\vdots & \vdots & \vdots & \vdots \\
0 \longrightarrow & 0 \longrightarrow & 0 \longrightarrow & 0 \\
\mathbb{Z} \longrightarrow & 0 \longrightarrow & \mathbb{Z} \longrightarrow & 0
\end{array}$$

Since it is clear that the maps are all zero the sequence has already both collapsed and degenerated at page 1, clearly taking cohomology gives the same thing, so E^2 and therefore the cohomology of $\mathbb{C}P^2$ is just $\mathbb{Z}$ in even degrees as seen here.

3 Hopf Fibration

The generalised Hopf fibration is given by

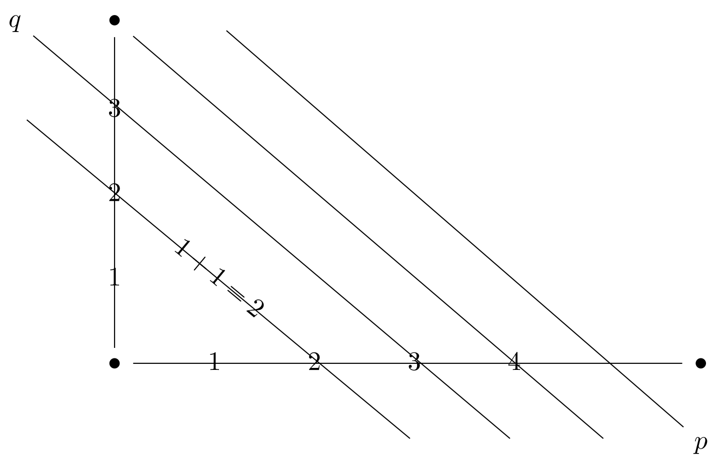
$$U(1) = S^1 \to S^{2n+1} \to \mathbb{C}P^n$$

Where the surjection is just the projection quotienting out the antipodal points. We also handle the case of $n = \infty$. In this case the E^1 page is hard to calculate. This is because it is not easy to find the preimage of the p skeleton, $\mathbb{C}P^p \subseteq \mathbb{C}P^n$, this is a sort of S^1 bundle over $\mathbb{C}P^p$ and non-trivial, as well as finding the relative cohomology groups. Instead we will just start at the E^2 page, and using the fact that the E_*^∞ groups are given by $H^*(S^{2n+1})$ (the cohomology of the total space, the associtated gradeds). Because these groups are known we will try to find the "difference" between these groups and the E^2 page and thereby deduce the E^2 page.The E^2 page is as before

$$\begin{array}{cccc}
\vdots & \vdots & \vdots & \vdots \\
H^0(\mathbb{C}P^n; H^1(S^1)) & H^1(\mathbb{C}P^n; H^1(S^1)) & H^2(\mathbb{C}P^n; H^1(S^1)) & H^3(\mathbb{C}P^n; H^1(S^1)) \\
H^0(\mathbb{C}P^n; H^0(S^1)) & H^1(\mathbb{C}P^n; H^0(S^1)) & H^2(\mathbb{C}P^n; H^0(S^1)) & H^3(\mathbb{C}P^n; H^0(S^1))
\end{array}$$

Becuase $H^i(S^1) = \mathbb{Z}\delta_{i=0,1}$ only the two bottom rows remain, the rest of the groups are zero. The differentials are going one down and two to the right. From this we see that on the E^3 page all differentials must be zero and hence the sequence stabilises on E^3, which in turn therefore is given by the cohomology of the total space S^{n+1}. Thus the E^3 page must be given by the associated graded of the cohomology of the sphere. The sphere has cohomology $H^i(S^{2n+1}) = \mathbb{Z}\delta_{i=0,2n+1}$.

Now to use this fact we need to understand how to go from the associated graded, or the E^∞ page, to the cohomology of the total space and vice versa. Heres how I was told how to do it. First look at the E^∞ page, then we will extract $H^n(E)$ from the diagonal $E^\infty_{p,q}$ where $p+q=n$



So fixing an n picks out a diagonal, this diagonal defines a collection of extension problems. Because we are cohomologically graded our arrows point down and to the right, so we write one of the diagonals as

$$0 \to A_1 \to A_2 \to \cdots \to A_k \to 0$$

note that it is bounded because our sequence is first quadrant. Now we have some groups, $A_1, ..., A_k$ and we we start with the extension problem

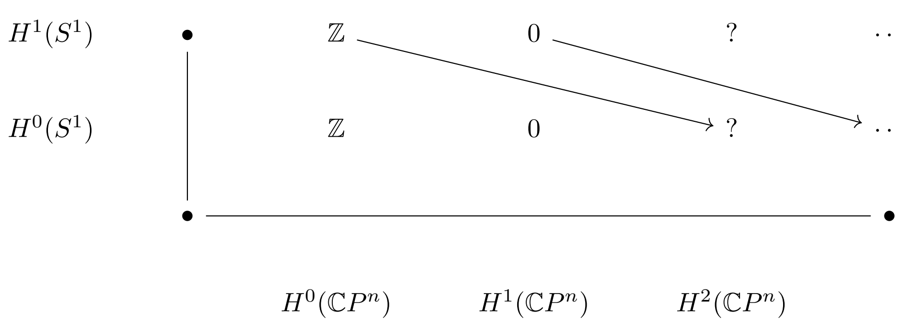
$$0 \to A_k \to E_1 \to A_{k-1} \to 0$$

then we can iterate, solving the next problem,

$$0 \to E_1 \to E_2 \to A_{k-2} \to 0$$

etc, because this is a bounded chain the process terminates and the group will be the required cohomology. **However** extension problems are not always unique, moreover the spectral sequence does not contain extra information to fix the solutions, thus we only have *possible* choices for the cohomology groups, as *some* solution. For us this wont matter because the extension problem will be unique.

Now we can apply this and our knowledge about the total cohomology to find the cohomology of the base. First we know only that on the E^∞ page all diagonals are zero execpt the zeroeth diagonal and the $2n+1$ diagonal on which there will apear at least a $\mathbb{Z}$ (there may be other torsion groups, that lead to trivial extensions!). Now we also know that our space is connected simply connected, in particular its zeroeth cohomology is $\mathbb{Z}$ and first cohomology is zero (see remark for details). So we start out knowing at least a couple groups on the E^2 page:



(Notive that the axis labelled can be tensored together to get the group in the middle, this is true by the universal coefficients in this case because some Tor terms vanish). We also know on the $E^3 = E^\infty$

page that there is a $\mathbb{Z}$ in the lower left corner, as the $\mathbb{Z}$ on that diagonal cannot appear elsewhere. The same goes for the $\mathbb{Z}$ in the $2n+1$ degree we know the diagonal and it must be contained to the bottom two rows (all other groups because they are cohomology of previous page must be zero, as all other groups are already zero). So we know the E^3 page looks like the following

$$
\begin{array}{c|cccccc}
\bullet & 0 & 0 & \cdots & {\color{red}?} & 0 & \cdots \\
 & \mathbb{Z} & 0 & \cdots & 0 & {\color{red}?} & \cdots \\
\hline
\bullet & & & & & {\color{red}2n+1} & \bullet
\end{array}
$$

where at least one of the ${\color{red}?}$ are $\mathbb{Z}$ (they might both be non-zero). Now we can solve the problem. Look at the $\mathbb{Z}$ at $E^1_{0,1}$, then on the next page it is zero, thus it is either in the image of the map in or not in the kernel of the map out. The map in is the zero map and so to get a zero in cohomology the kernel must be zero, hence we have an injection. Thus the group $E^1_{2,0}$ must contain a $\mathbb{Z}$, but the cohomology on the next page is again a zero and moreover we know that the kernel of the map out is the whole group (its the zero map) and therefore the map in must be everything and so the map is a surjection. Thus the map $E^1_{0,1} \to E^1_{2,0}$ is an isomorphism!

$$
\begin{tikzcd}
H^1(S^1) & \bullet & \mathbb{Z} \arrow[drr, "\sim"] & 0 \arrow[drr] & \mathbb{Z} & 0 & ? & \cdots \\
H^0(S^1) & & \mathbb{Z} & 0 & \mathbb{Z} & 0 & ? & \cdots \\
 & \bullet & & & & & & \bullet \\
 & & H^0(\mathbb{C}P^n) & H^1(\mathbb{C}P^n) & H^2(\mathbb{C}P^n) & & &
\end{tikzcd}
$$

Similarly the first zero column implies that the third column must be zero. Note the groups in the second row are isomorphic to those in the first row from the setup and thats how we can raise the group to the second row. This process clearly can be iterated (indefinitely for $\mathbb{C}P^\infty$) up until the last $2n+1$ diagonal where we again need to do some analysis, but up to that point we have already shown that the cohomology groups alternate zero and $\mathbb{Z}$.

Remark. (Jayden) The general philsophy in spectral sequences is that there are two directions

$$E^1 \leftrightarrow E^\infty$$

One can start at the first page and compute the later ones, or start at the last page and compute the earlier ones. In general starting at the last page will require only formal homological manipulations, while going forward will involve more in depth computations around differentials.

Remark. Simply connected spaces have trivial zeroth and first cohomology. This can be deduced from the vanishing of Tor terms in the universal coefficients theorem. Another way of seeeing this is that these groups are already stable on the E^2 page, this is clear because both differentials are zero, and on the next page in this case the group must be zero, as it is the cohomology of the sphere.

3.1 Frame Bundles

Consider the fibration

$$\mathrm{SO}(2) = S^1 \to \mathrm{SO}(3) \to S^2$$

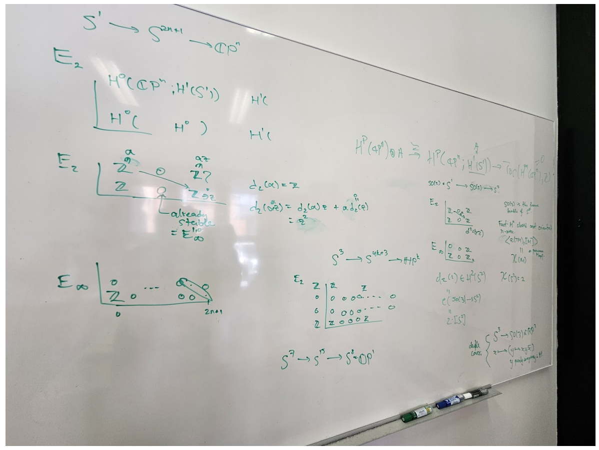


Figure 1: A deduction of the ring structure on the LHS.

The E^2 page is now the cohomology of the base, which is S^2, which we know and therefore can write out straight away

$$\begin{tikzcd} \mathbb{Z} \arrow[drr, "d"] & 0 & \mathbb{Z} \\ \mathbb{Z} & 0 & \mathbb{Z} \end{tikzcd}$$

there is a single differential that can be non-trivial.

3.2 Classifying Space of Unitary Groups

There is a fibration

$$U(n) \to EU(n) \to BU(n)$$

for which the $n = 1$ case is actually given by the sphere bundle over $\mathbb{C}P^\infty$ above.

3.3 Classifying Spaces Compared

There is a ses of groups

$$SU(2) \to U(2) \xrightarrow{det} S^1$$

and the classyfing space functor sends that to a fibration

$$BSU(2) \to BU(2) \to BS^1 = \mathbb{C}P^\infty$$

This collapses using the so called “checker board principle”.

References